TUTORIAL NOTES FOR MATH4220

JUNHAO ZHANG

1. MEAN-VALUE PROPERTY AND HARMONIC FUNCTIONS

Let us recall the Mean-value property of the harmonic functions.

Theorem 1 (Mean-value property). *The function u is harmonic in $\Omega \subset \mathbb{R}^n$ if and only if for any $B_r(x)$ with $\overline{B_r(x)} \subset \Omega$,*

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)dS_y.$$

Using the Mean-value property, we can investigate various properties of harmonic functions.

Example 2. Suppose u is a harmonic in $\Omega \subset \mathbb{R}^n$, prove that for arbitrary $\overline{B_r(x)} \subset \Omega$,

$$|Du(x)| \leq \frac{2^{n+1}n}{r^{n+|\alpha|}|B_1|}\|u\|_{L^1(B_r(x))}.$$

In general, for multi-index $\alpha = (\alpha_1, ..., \alpha_n)$,

$$|D^\alpha u(x)| \leq \frac{C_{|\alpha|}}{r^{n+|\alpha|}}\|u\|_{L^1(B_r(x))},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$C_{|\alpha|} = \begin{cases} \frac{1}{|B_1|}, & |\alpha| = 0, \\ \frac{(2^{n+1}n|\alpha|)^{|\alpha|}}{|B_1|}, & |\alpha| = 1, ... \end{cases}.$$

Proof. The case for $|\alpha| = 0$ is a direct result of Mean-value property.

For $|\alpha| = 1$, by the Mean-value property,

$$\begin{aligned} |\partial_i u(x)| &= \left|\frac{2^n}{r^n|B_1|}\int_{B_{\frac{r}{2}}(x)} \partial_i u(y)dy\right| \\ &\leq \left|\frac{2^n}{r^n|B_1|}\int_{\partial B_{\frac{r}{2}}(x)} u(y)\cdot n_i(y)dS_y\right| \\ &\leq \frac{2n}{r}\|u\|_{L^\infty(\partial B_{\frac{r}{2}}(x))}. \end{aligned}$$

For $y \in \partial B_{\frac{r}{2}}(x)$, we have $B_{\frac{r}{2}}(y) \subset B_r(x)$, then by the Mean-value property again,

$$\|u\|_{L^\infty(\partial B_{\frac{r}{2}}(x))} \leq \frac{2^n}{r^n|B_1|}\|u\|_{L^1(B_r(x))},$$

therefore

$$|\partial_i u(x)| \leq \frac{2^{n+1}n}{r^{n+1}|B_1|}\|u\|_{L^1(B_r(x))}.$$

For $|\alpha| \geq 2$, we prove the result by induction. Suppose for $|\alpha| \leq k-1$, it is valid that

$$|D^\alpha u(x)| \leq \frac{C_{|\alpha|}}{r^{n+|\alpha|}}\|u\|_{L^1(B_r(x))},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$C_{|\alpha|} = \begin{cases} \frac{1}{|B_1|}, & |\alpha| = 0, \\ \frac{(2^{n+1}n|\alpha|)^{|\alpha|}}{|B_1|}, & |\alpha| = 1, ... \end{cases}$$

for arbitrary $\overline{B_r(x)} \subset \Omega$. Now for arbitrary multi-index α with $|\alpha| = k$, there exist some $i \in \{1, ..., n\}$ and multi-index β with $|\beta| = k-1$ such that

$$D^\alpha u = \partial_i D^\beta u,$$

then by the Mean-value property,

$$\begin{aligned} |\partial_i D^\beta u(x)| &= \left| \frac{k^n}{r^n|B_1|} \int_{B_{\frac{r}{k}}(x)} \partial_i D^\beta u(y) dy \right| \\ &\leq \left| \frac{k^n}{r^n|B_1|} \int_{\partial B_{\frac{r}{k}}(x)} D^\beta u(y) \cdot n_i(y) dS_y \right| \\ &\leq \frac{nk}{r}\|D^\beta u\|_{L^\infty(\partial B_{\frac{r}{k}}(x))}. \end{aligned}$$

For $y \in \partial B(x, \frac{r}{k})$, we have $B_{\frac{k-1}{k}r}(y) \subset B(x, r)$, then by the Mean-value property again,

$$\|D^\beta u\|_{L^\infty(\partial B_{\frac{r}{k}}(x))} \leq \frac{[2^{n+1}n(k-1)]^{k-1}}{\left(\frac{k-1}{k}r\right)^n |B_1|}\|u\|_{L^1(B_r(x))},$$

therefore

$$|D^\alpha u(x)| \leq \frac{(2^{n+1}nk)^k}{r^{n+k}|B_1|}\|u\|_{L^1(B_r(x))}.$$

□

Example 3. Assume u is harmonic in Ω. Prove that u is analytic in Ω.

Proof. To prove u is analytic, we show that u can be represented by a convergent power series in a neighborhood of arbitrary $x_0 \in \Omega$.

Denote

$$r := \frac{1}{4}\text{dist}(x_0, \partial\Omega), \quad M := \frac{1}{r^n|B_1|}\|u\|_{L^1(B_{2r}(x_0))}.$$

For each $x \in B_r(x_0)$, we have $B_r(x) \subset B_{2r}(x_0) \subset \Omega$, then

$$\|D^\alpha u\|_{L^\infty(B_r(x_0))} \leq M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|}|\alpha|^{|\alpha|}.$$

Since

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|}n^{|\alpha|}\alpha!,$$

therefore

$$\|D^\alpha u\|_{L^\infty(B_r(x_0))} \leq CM\left(\frac{2^{n+1}n^2e}{r}\right)^{|\alpha|}\alpha!.$$

Consider the Taylor series for u at x_0,

$$S_N(x) = \sum_{|\alpha|\leq N-1} \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha,$$

we claim that $S_N(x)$ converges provided $x \in B_\varepsilon(x_0)$ with $0 < \varepsilon < \frac{r}{2^{n+2}n^3 e}$. Indeed, denote the remainder term

$$R_N(x) := u(x) - S_N(x) = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x-x_0))(x-x_0)^\alpha}{\alpha!},$$

for some $0 \leq t \leq 1$. Then for $x \in B_\varepsilon(x_0)$ with $0 < \varepsilon < \frac{r}{2^{n+2}n^3 e}$,

$$\begin{aligned} |R_N(x)| \leq & CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2 e}{r}\right)^N \left(\frac{r}{2^{n+2}n^3 e}\right)^N \\ \leq & CMn^N \frac{1}{(2n)^N}, \end{aligned}$$

therefore by letting N goes to infinity, we have

$$\lim_{N\to\infty} |R_N(x)| = 0,$$

which implies that $S_N(x)$ converges to $u(x)$. □

A Supplementary Problem

Problem 4. Suppose u is harmonic and positive in $\Omega \subset \mathbb{R}^n$. Let $x_0 \in \partial\Omega$ be a zero of u. If there exists a ball $B_r(x) \subset \Omega$ such that $\partial\Omega \cap \overline{B_r(x)} = \{x_0\}$, then

$$\frac{\partial u}{\partial v}(x_0) > 0,$$

where $v = \frac{x-x_0}{r}$. In particular, if $\partial\Omega$ is C^1 at x_0, then v is the inward unit normal to $\partial\Omega$ at the point x_0.

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

Email address: `jhzhang@math.cuhk.edu.hk`